Fourier Methods

9TH NOVEMBER 2020

Outline

- Fourier Series
- Application of Fourier Sine Series to a Triangular Function
- Application to the Energy in the Normal Modes of a Vibrating String
- Fourier Series Analysis of a Rectangular Velocity Pulse on String
- The Spectrum of a Fourier Series
- Fourier series to Fourier Integral
- Fourier Transform

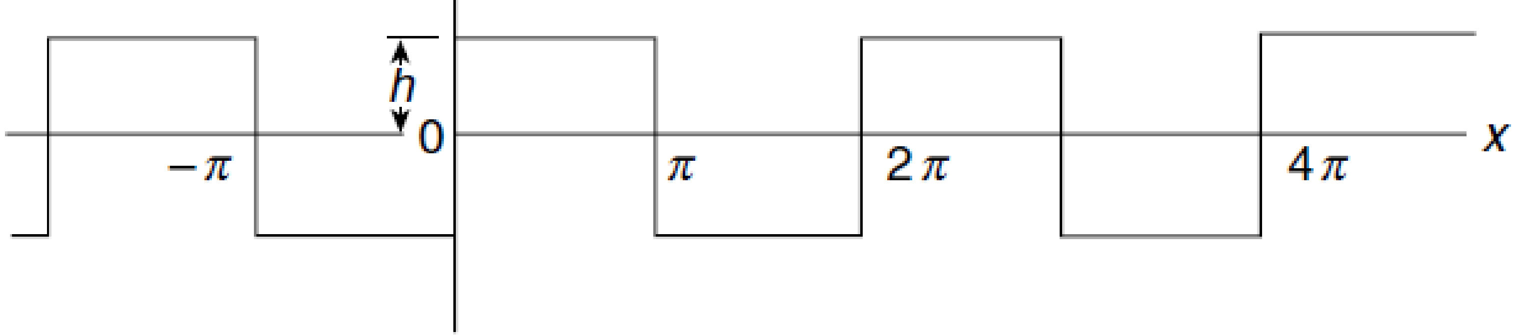
Fourier Series

- Any function which repeats itself regularly over a given interval of space or time is called **a periodic function**; e.g. $f(x) = f(x \pm \alpha)$ where α is the interval or period.
- Almost all periodic functions of interest using the method of Fourier Series may be represented by the series,

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx$$

Fourier series of a square wave

$$f(x) = \frac{4h}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x \ldots\right)$$



Fourier Series in several equivalent forms

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos(nx - \theta_n)$$

Fourier coefficients : $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$\mathrm{b}_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

where $c_n^2 = a_n^2 + b_n^2$ and $\tan\theta_n = b_n / a_n$

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx}$$

where $2d_n = a_n - ib_n \ (n \geq 0)$ and $2d_n = a_{-n} + ib_{-n} \ (n < 0)$

Coefficients of a_n and b_n

- To find the values of the coefficient either a_n or b_n, let's multiply the following equation with $\cos mx$ or $sin\, mx$ and then integrate with respect to x over the period of 0 to 2π

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\, x + b_n \sin n\, x)$$

- And then apply the following conditions:

$$\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} 0\ if\ m \neq n \\ \pi\ if\ m = n \end{cases}$$

$$\int_0^{2\pi} sin\, mx \cos nx\, dx = 0\ for\ all\ m\ and\ n$$

Now try to determine either a_n or b_n

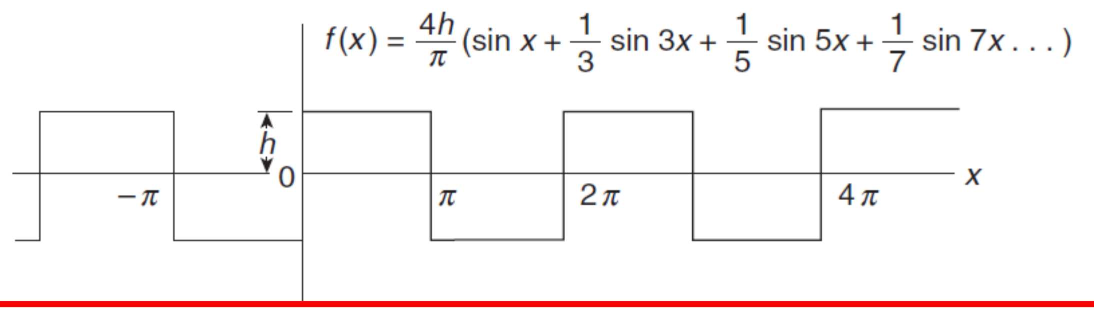
Interpretation of the Fourier Series (1)

1) The constant term $(1/2)a_0$: this is the average of the function over an interval.

Recall the coefficient a_n : $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

As $n = 0$, we have $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$ and this leads to $\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$ which is the average of the function over 2π interval .

The constant term can be varied by moving the function with respect to the x axis.



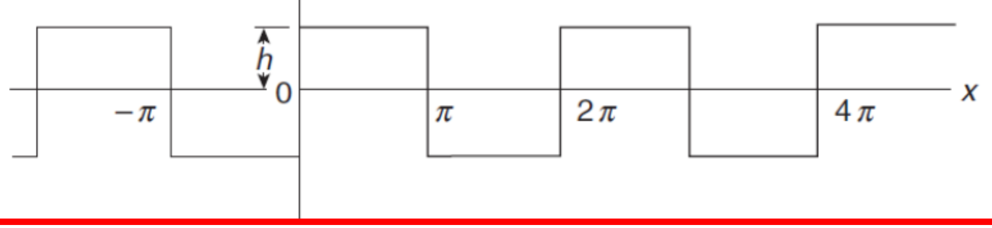
The periodic function is symmetric about x axis, its average $\frac{a_0}{2}$ is zero.

Interpretation of the Fourier Series (2)

2) Generally, Fourier series is represented by the combination of even and odd parts. Whether the function is completely even (f(x) = f(-x)) or completely odd (f(x) = -f(-x)) can often determined by the position of the y-axis.

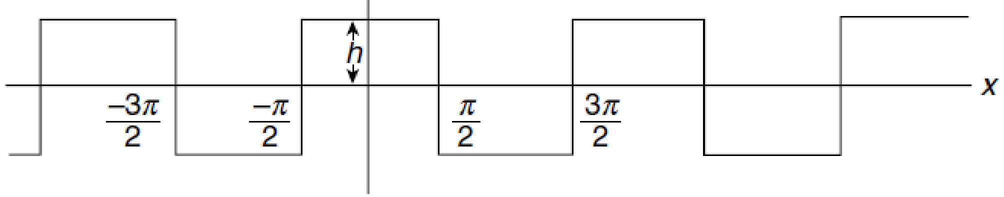
For example

$$f(x) = \frac{4h}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x \ldots\right)$$



This periodic function is represented by **odd function** with zero constant.

$$f(x) = \frac{4h}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x \ldots\right)$$

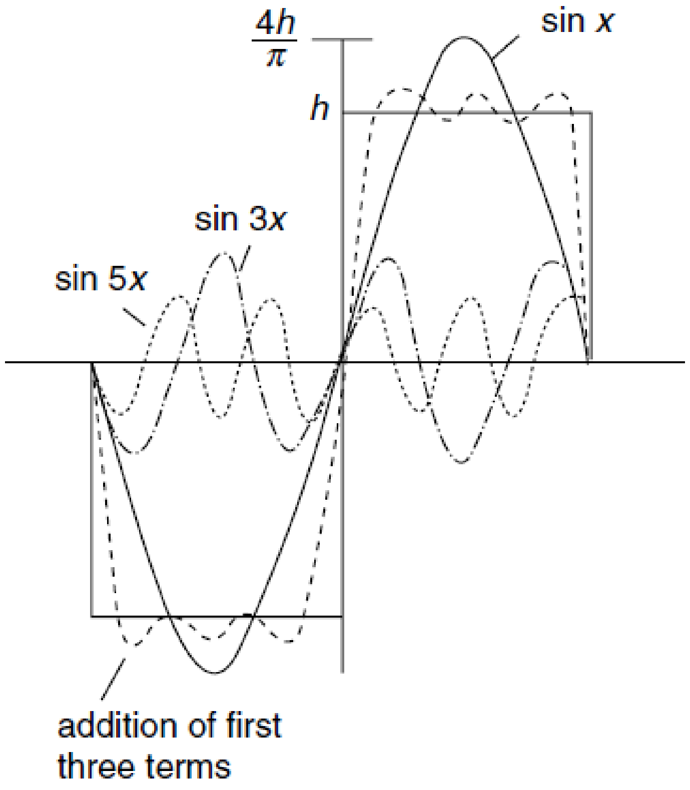


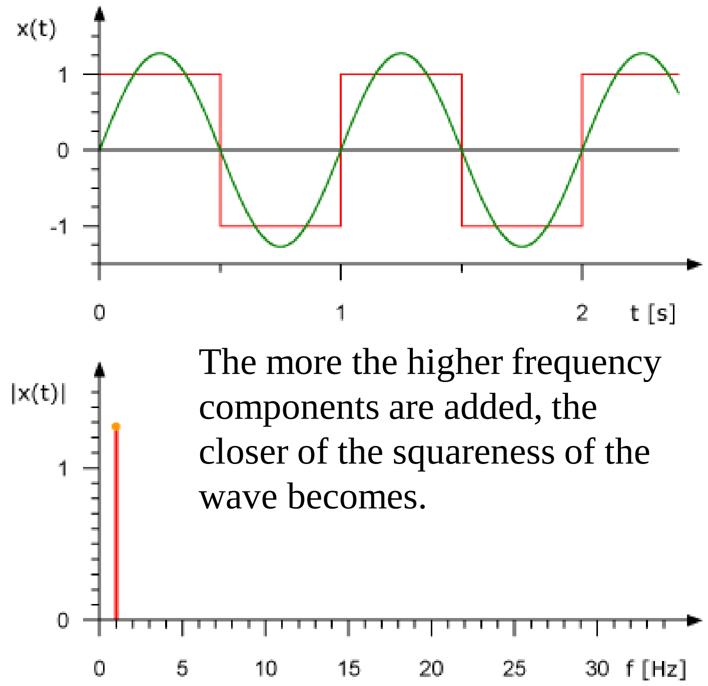
This periodic function is represented by **even function** with zero constant.

Interpretation of the Fourier Series (3)

3) Terms in the series and their combination are responsible for the shape of the series representation.

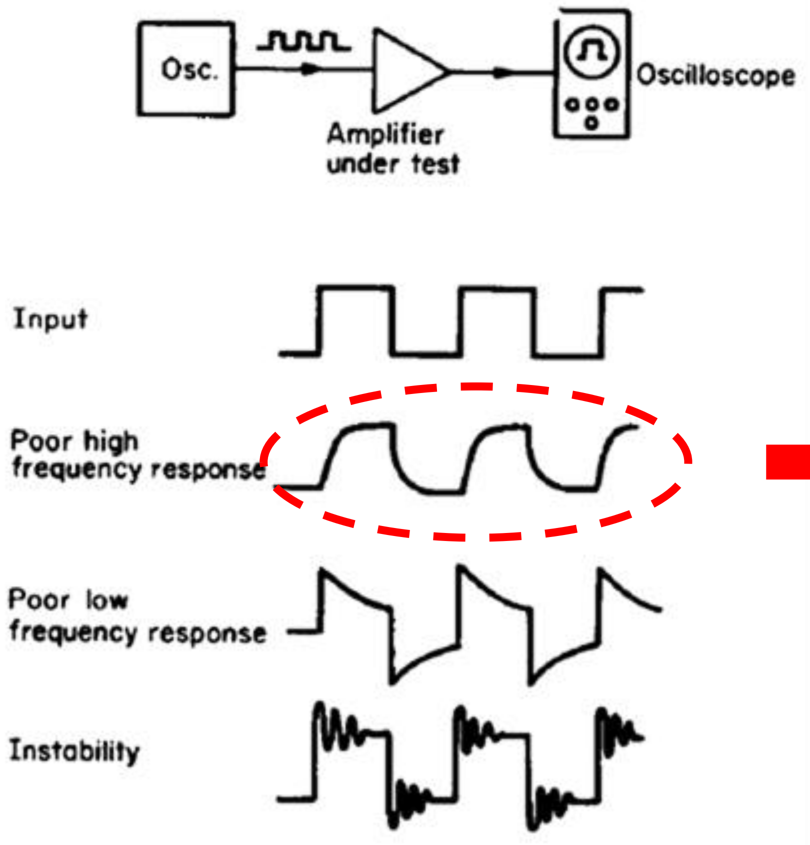
- The fundamental or first harmonic has the frequency of the square wave.
- The higher frequencies build up the squareness of the waves.
- The highest frequencies are responsible for the sharpness of the vertical sides of the waves.





The more the higher frequency components are added, the closer of the squareness of the wave becomes.

Frequency response test of amplifiers

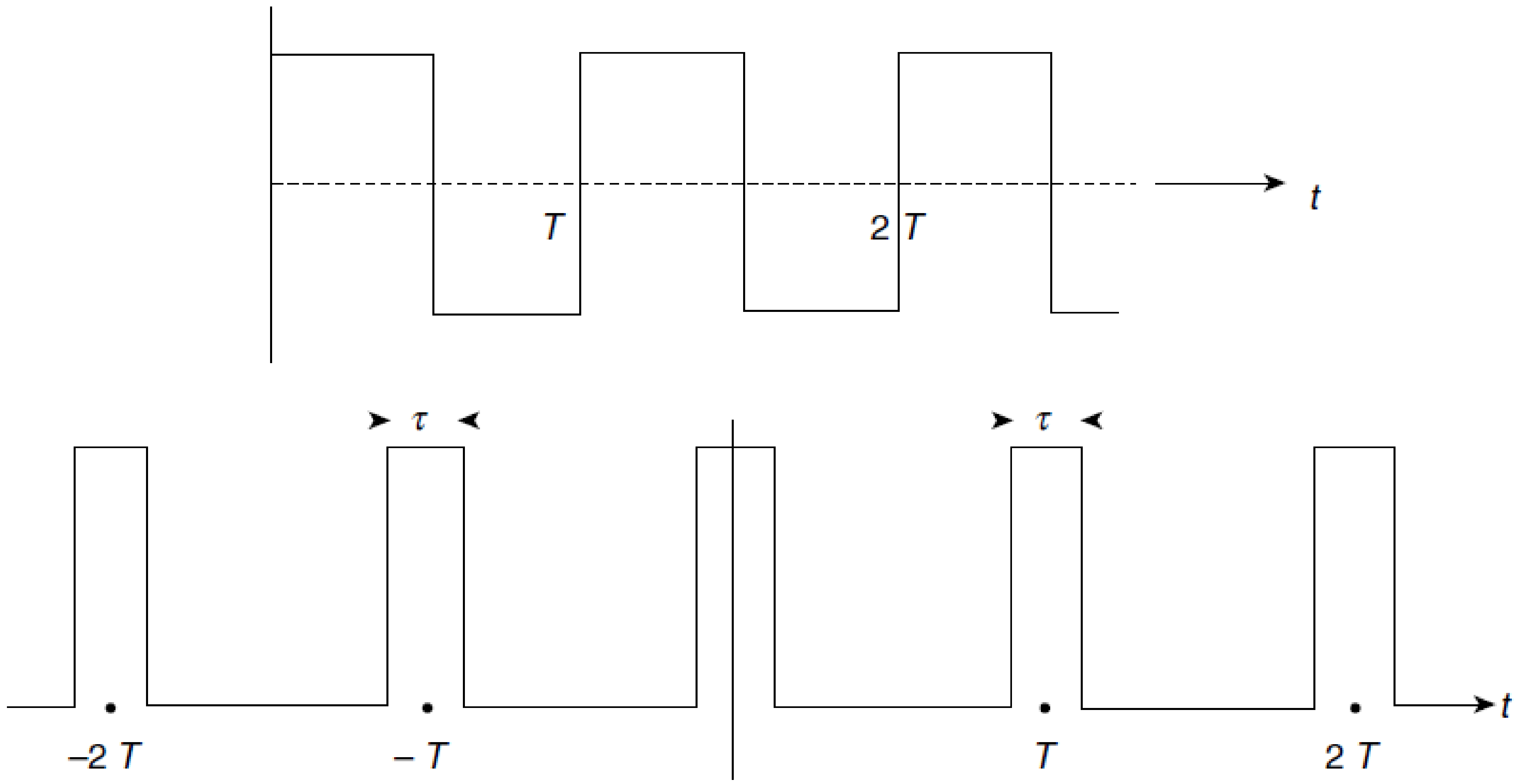


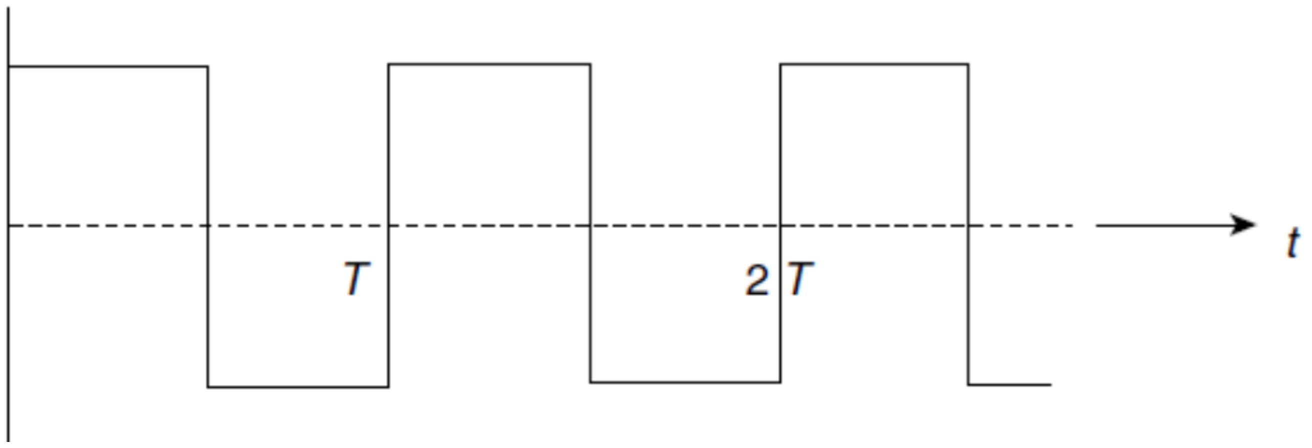
Square wave oscillator used to test an amplifier

Loss of the sharpness at the edges of the waves shows that the amplifier response is limited at the higher frequency range.

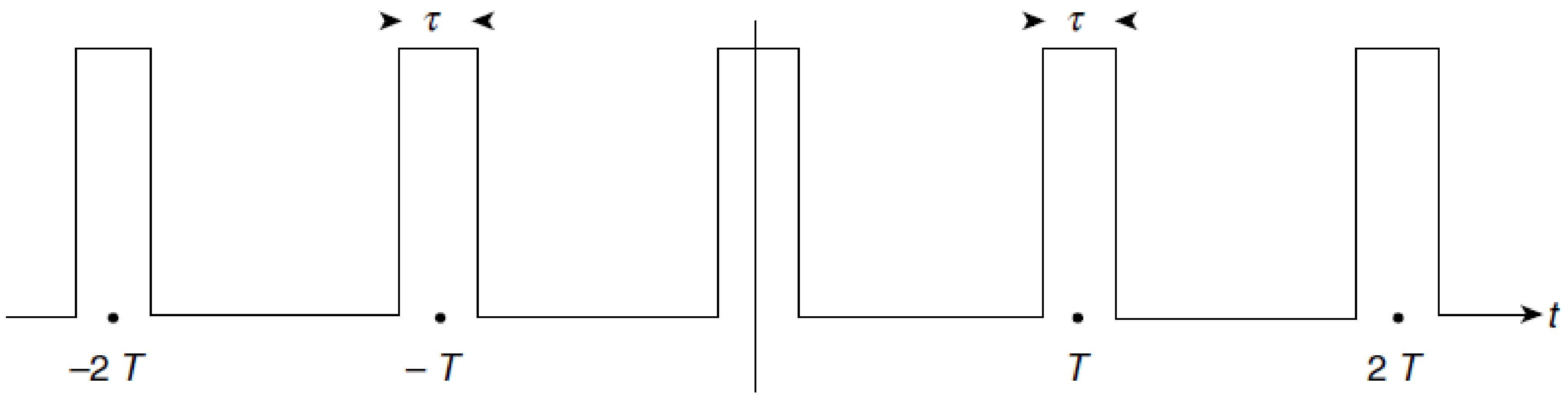
Problem 10.1

After inspection of the two wave forms in the diagram what can you say about the values of the constant, absence or presence of sine terms, cosine terms, odd or even harmonics, and range of harmonics required in their Fourier series representation? (Do not use any mathematics.)





- Constant is zero (symmetry around x axis).
- Represented by odd function (sine function).
- More than 3 harmonics required to faithfully represent the edge sharpness.



- Constant is not zero (asymmetry around x axis).
- Represented by even and odd functions.
- More than 3 harmonics required to faithfully represent the edge sharpness.

Fourier Series for any interval

- Any section or interval of a well behaved function may be chosen and expressed in terms of Fourier Series.
- This series will accurately represent the function only within the chosen interval.
- If the interval is represented by a Fourier cosine series the repetition will be that of an even function, if the representation is a Fourier sine series and odd function repetition will follow.

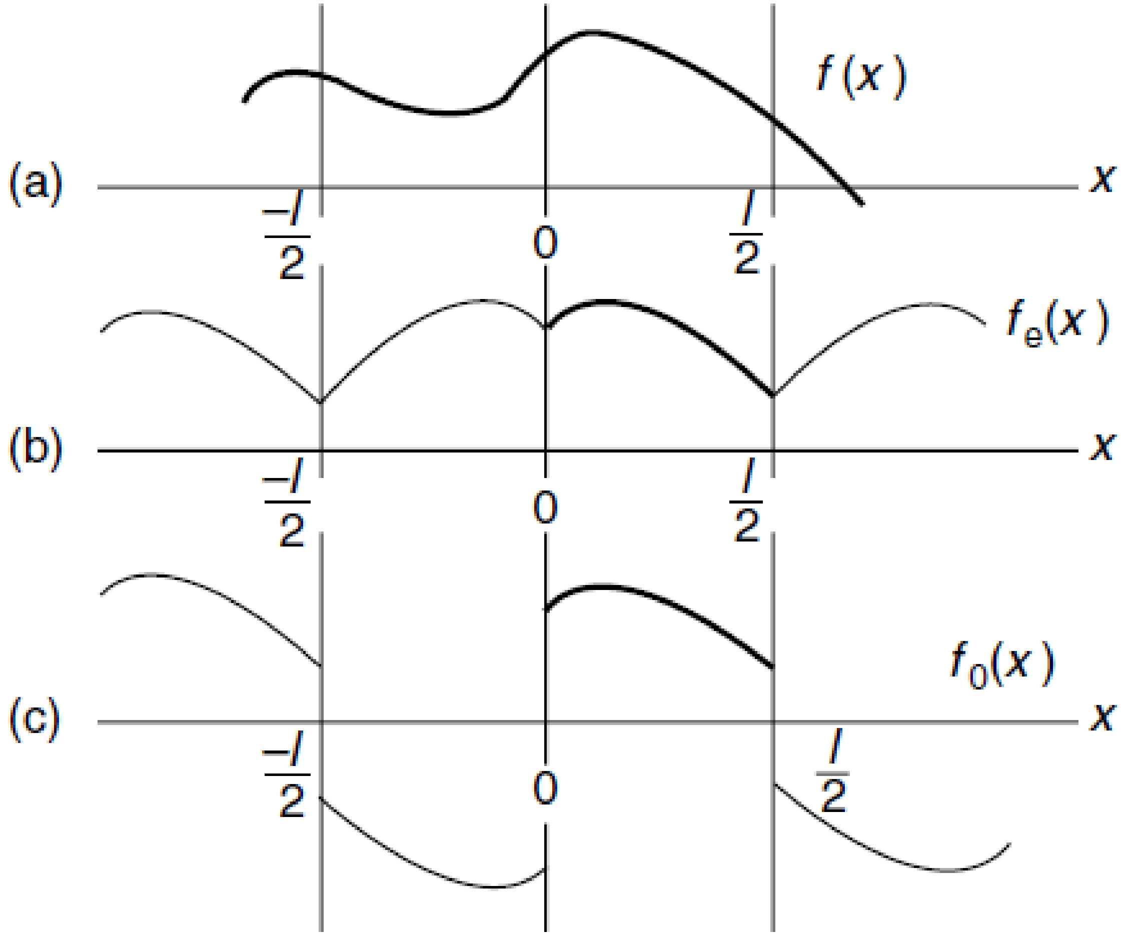
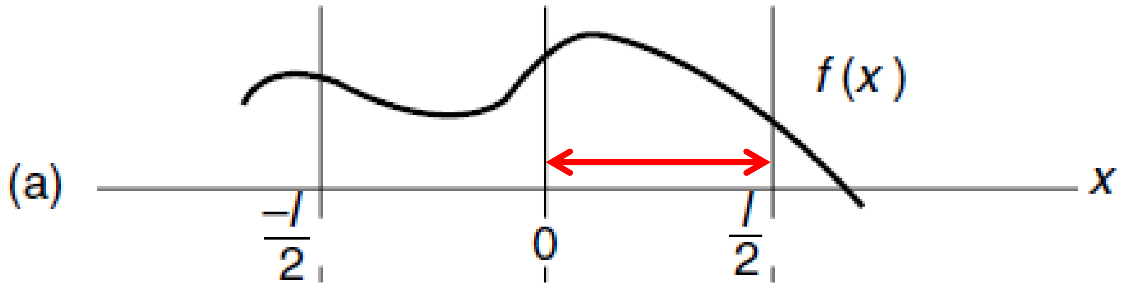


Figure 10.4 A Fourier series may represent a function over a selected half-interval. The general function in (a) is represented in the half-interval $0 < x < l/2$ by f_e, an even function cosine series in (b), and by f_o, an odd function sine series in (c). These representations are valid only in the specified half-interval. Their behaviour outside that half-interval is purely repetitive and departs from the original function

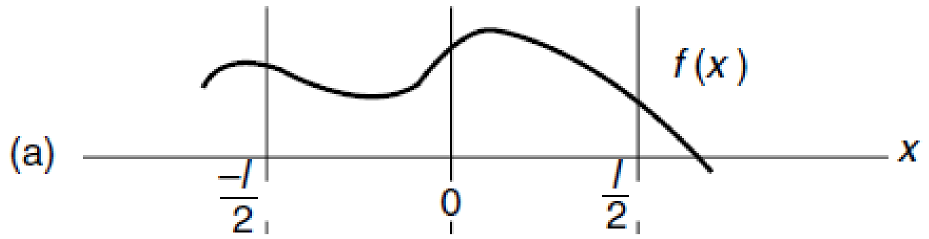
Arguments in terms of phase angles

- So far the arguments of cosine and sine functions are given as x assumed to be measured in radians.
- However, if x is assigned as a **distance** the a factor of $2\pi/l$ is needed to transform the distance into an angle unit. With this factor, when $x = l$, the phase angle changes by 2π.
- For example, an original function shown in the figure below can be represented as a sine series (odd function) over a half interval [0,l/2]



$$f\ (x) = \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{l} x$$

- The given function can be represented by the sine series as follows,



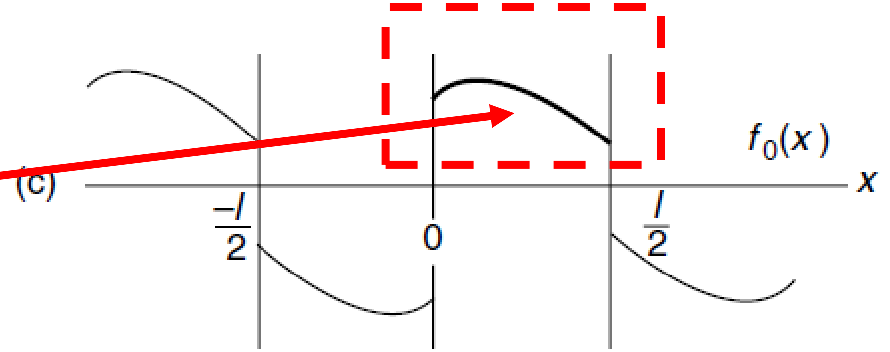
$$f(x) = f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n}{l}x$$

$$b_n = \frac{1}{\frac{1}{2}\text{interval}}\int_{-l/2}^{l/2} f(x)\sin\frac{2\pi nx}{l}dx$$

$$= \frac{2}{l}\left[\int_{-l/2}^{0} f_o(x)\sin\frac{2\pi nx}{l}dx + \int_{0}^{l/2} f_o(x)\sin\frac{2\pi nx}{l}dx\right]$$

$$= \frac{4}{l}\int_{0}^{l/2} f\ (x)\sin\frac{2\pi nx}{l}dx$$

$$\therefore f(x) = f_o(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{l}dx$$



Note : only the behavior of the function within the half interval 0 to l/2 can be described by $f_o(x)$

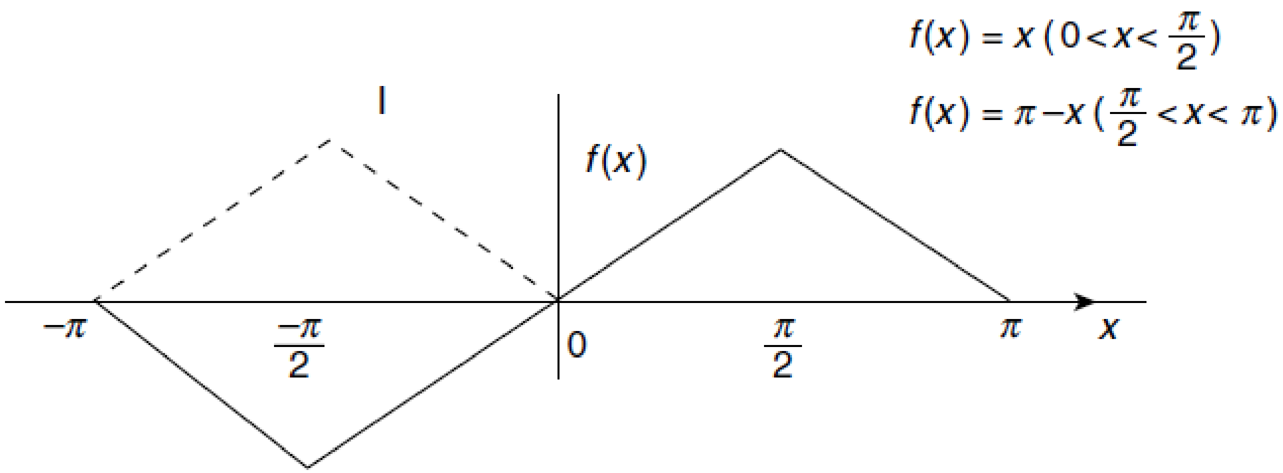
Q: Application of Fourier Sine Series to a Triangular Function

- The figure below shows a function which we are going to describe by a sine series in half-interval 0 to π. The function is

$$f(x) = x \quad \left(0 < x < \frac{\pi}{2}\right)$$

and $$f(x) = \pi - x \quad \left(\frac{\pi}{2} < x < \pi\right)$$

This function defined over a limited interval can be used to described the behavior of a plucked string.



A: Application of Fourier Sine Series to a Triangular Function

Writing $f(x) = \sum b_n \sin nx$ gives

Fourier sine series is chosen to represent the half interval of a triangular function

b_n : coefficients of the odd part

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, \mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx \, \mathrm{d}x$$

$$= \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

When n is even $\sin n\pi/2 = 0$, so that only terms with odd values of n are present and

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\right)$$

Permitted harmonics of the plucked string

Note that at $x = \pi/2$, $f(x) = \pi/2$, giving

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

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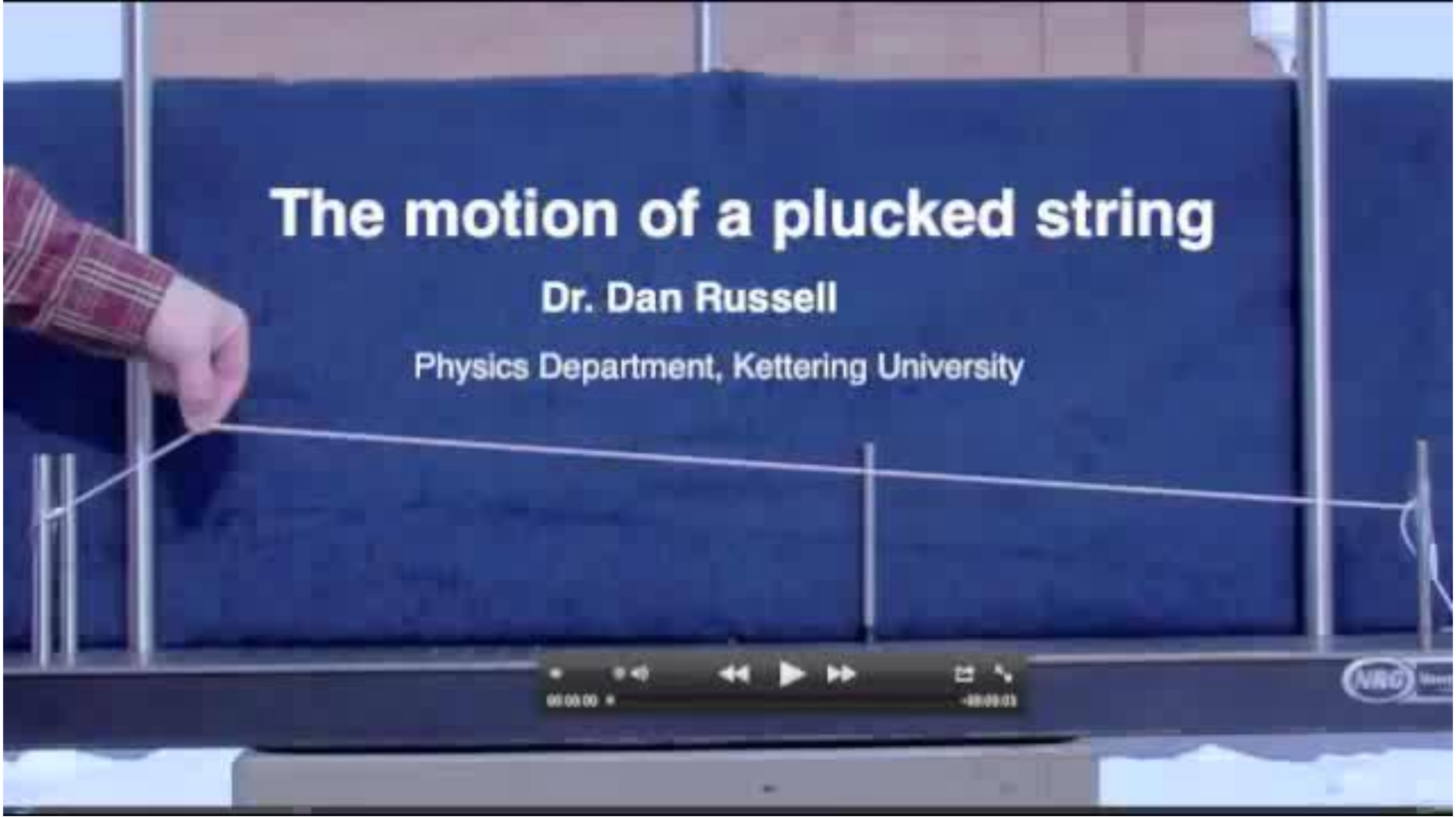
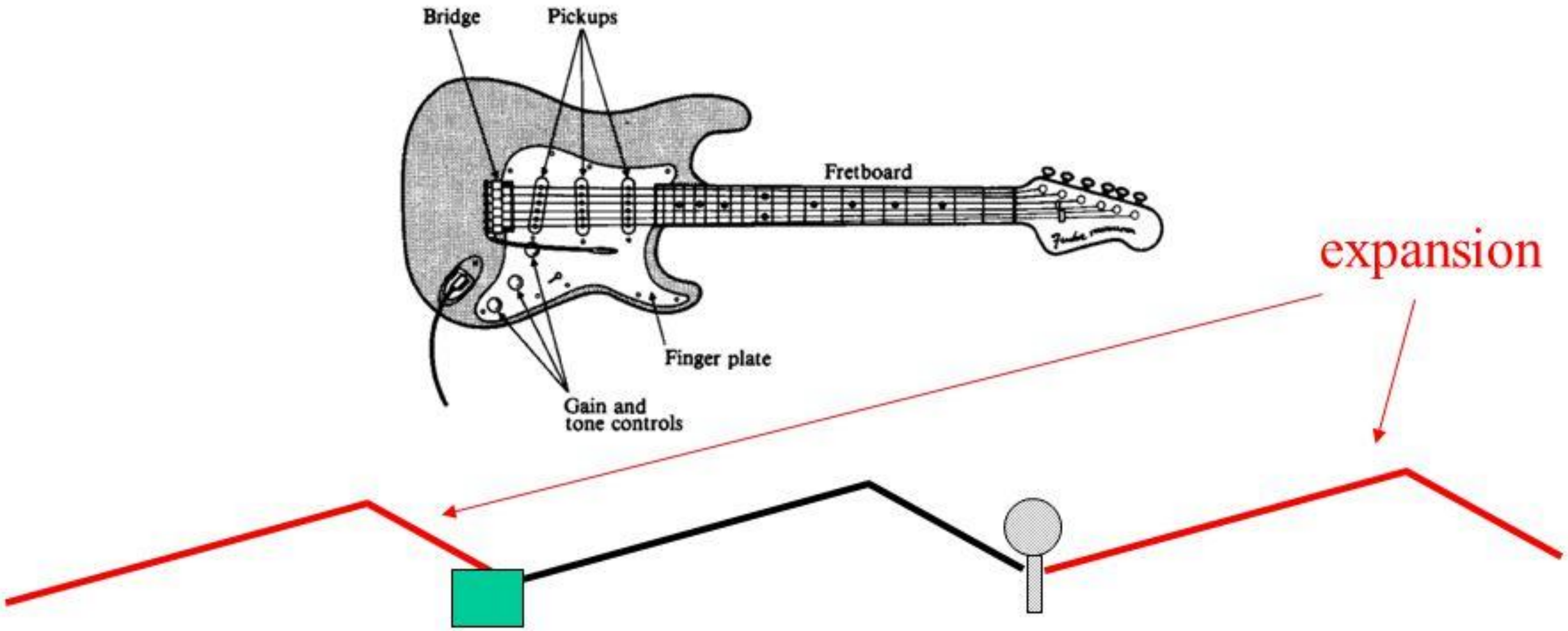


https://www.youtube.com/watch?v=_X72on6CSL0

Half Range Expansion

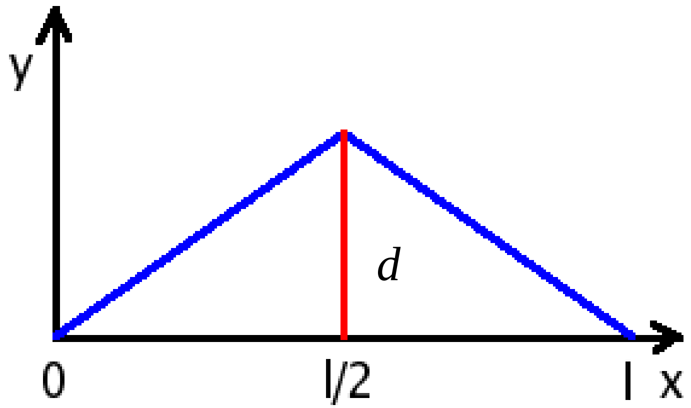
Expansion is useful when a function is defined only on a given interval, say between 0 and L. This situation is very common in real life: For example, the vibration of a guitar string occurs only between its bridge and tension peg.



http://slideplayer.com/slide/6018598/20/images/5/Half+Range+Expansion.jpg

Application to the energy in the normal modes of a vibrating string

- If we take a string of length *l* with fixed end and pluck its center a distance *d*, we have a triangular configuration of the a interval as shown in the figure below.
- According to the previous section, this configuration expected to be represented by the **Fourier sine series.**



http://users.aber.ac.uk/ruw/teach/260/wave2.php

- Under this particular situation, the series representing the plucked string at time *t* is equivalent to the total displacement of the normal modes or standing waves occurring once the string is released.

- To find the displacement of a normal mode or standing wave vibration, the following standing wave equation has to be solved using the method of separation of variable.

- The wave equation : $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

- The solution that fits the boundary condition; i.e., $y(x = 0, t) = y(x = l, t) = 0$, is found to be

$$y_n = \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \sin \frac{\omega_n x}{c}$$

Derive this!

Derivation of a normal mode wave function in summary

- Possible solution may be given by   y(x,t) = X(x).T(t)
- By substituting the solution back into the wave equation, rearrange the equation and equate each side of the equation to a constant, we then end up with two linear ODE as follows,

$$\frac{d^2X}{dx^2} + k^2X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + v^2k^2T = 0$$

- This leads to

$$X(x) = C_1 \cos kx + C_2 \sin kx$$

$$T(t) = D_1 \cos kvt + D_2 \sin kvt$$

Expression of a normal mode wave function

- From the derived general forms of the normal mode wave function, a particular form can be obtained once the boundary condition, i.e. $y(0,t) = y(l,t) = 0$ is considered.

$$y(x,t) = X(x) \cdot T(t) = (C_1 \cos kx + C_2 \sin kx) \cdot (D_1 \cos kvt + D_2 \sin kvt)$$

***Apply y*(0,*t*) = 0**

$$y(x,t) = X(x) \cdot T(t) = (C_2 \sin kx) \cdot (D_1 \cos kvt + D_2 \sin kvt)$$

***Apply y*(*l*,*t*) = 0**

For a normal mode n, the expression may be written as

$$y_n(x,t) = (A_n \cos\omega_n t + B_n \sin \omega_n t)\left(\sin \frac{\omega_n x}{c}\right)$$

, where $v = c = \omega/k$ and $\omega_n = n\pi c/l$

- The total displacement, which represents the shape of the plucked string at t, is given by summing the normal modes

$$y = \sum y_n = \sum \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right) \sin \frac{\omega_n x}{c}$$

- The arbitrary constants A_n and B_n have to be determined.
- Recall the initial conditions at t = 0; i.e., $y \neq 0$ and $v = dy/dt = 0$ **(stationary plucked string)**
- To obtain the representation of the series that satisfies the initial condition at t = 0, the total displacement is rewritten as

$$y_0(x) = \sum y_n(x) = \sum (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c}$$
$$= \boxed{\sum A_n \sin \frac{\omega_n x}{c}} \quad \text{at} \quad t = 0$$

Fourier series of displacement

- The velocity of the string at time t = 0.

$$v_0(x) = \frac{\partial}{\partial t} y_0(x) = \sum \dot{y}_n(x)$$

$$= \sum (-\omega_n A_n \sin \omega_n t + \omega_n B_n \cos \omega_n t) \sin \frac{\omega_n x}{c}$$

Fourier series of velocity ➡ $$= \boxed{\sum \omega_n B_n \sin \frac{\omega_n x}{c}} \quad \text{at} \quad t = 0$$

- **Both displacement $y_0(x)$ and velocity $v_0(x)$ at t = 0 can be written in the form of Fourier sine series.**
- Generally, A_n and B_n can be determined for a string of length l from

$$A_n = \frac{2}{l} \int_0^l y_0(x) \sin \frac{\omega_n x}{c} \, dx$$

⬅ **Derive this!**

$$\omega_n B_n = \frac{2}{l} \int_0^l v_0(x) \sin \frac{\omega_n x}{c} \, dx$$

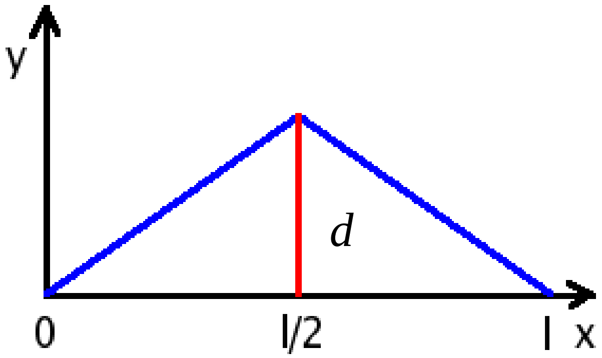
Determination of the Fourier coefficients

- Fourier coefficient A_n of displacement $y_0(x)$,

Similarly, the Fourier coefficient $\omega_n B_n$ can be obtained.

TRY THIS!

- To obtain particular forms of A_n or B_n, the displacement function of the plucked string has to be clearly stated.
- Referring to the figure, the displacement of a string length l, center plucked a distance d is given by



$$y_0(x) = \frac{2dx}{l} \qquad 0 \le x \le \frac{l}{2}$$
$$= \frac{2d(l-x)}{l} \qquad \frac{l}{2} \le x \le l$$

- Due to the plucked string released from rest ($v_0(x) = 0$), **B_n's are zero**.
- Only A_n's are of interest and found to be

$$A_n = \frac{2}{l}\left[\int_0^{l/2} \frac{2dx}{l} \sin\frac{\omega_n x}{c}\,\mathrm{d}x + \int_{l/2}^{l} \frac{2d(l-x)}{l} \sin\frac{\omega_n x}{c}\,\mathrm{d}x\right]$$
$$= \frac{8d}{n^2\pi^2} \sin\frac{n\pi}{2} \left(\text{for } \omega_n = \frac{n\pi c}{l}\right)$$

Only n odd is considered.

- Therefore, the total displacement can be written as $$y_0(x) = \sum_{n \text{ odd}} A_n \sin\frac{\omega_n x}{c} = \sum_{n \text{ odd}} \left(\frac{8d}{n^2\pi^2} \sin\frac{n\pi}{2}\right) \sin\frac{\omega_n x}{c}$$

- Recall the energy in each of normal modes of standing wave vibration, $\boxed{E_n = \frac{1}{4} m\omega_n^2 (A_n^2 + B_n^2)}$

- **Because B_n's are zero**, the energy of *n*th mode of vibration can be written as

$$\because \omega_n = n\pi c/l$$
$$\therefore E \propto n^{-2}$$

$$E_n = \frac{1}{4} m\omega_n^2 A_n^2 = \frac{64 d^2 m\omega_n^2}{4(n^2\pi^2)^2}$$

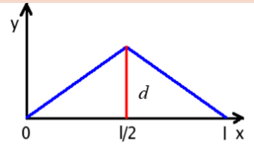
- The total vibrational energy of the string is given by $E = \sum_{n\,\mathrm{odd}} E_n = \frac{16d^2m}{\pi^4} \sum_{n\,\mathrm{odd}} \frac{\omega_n^2}{n^4} = \frac{16d^2c^2m}{\pi^2 l^2} \sum_{n\,\mathrm{odd}} \frac{1}{n^2}$

- Or $E = \sum E_n = \frac{2mc^2d^2}{l^2} = \frac{2Td^2}{l}$ where $\sum_{n\,\mathrm{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$

From *** in p.17

In summary,
- **A plucked string can be represented by a Fourier sine series.**
- **In the absence of dissipation, the total vibration energy must equal to the potential energy of the plucked string before release.**
- **The energy *n*th mode is proportional to n^{-2}. Higher modes contribute less energy to the vibration.**
- **Even modes are forbidden by the initial boundary conditions**

Summary

(1) General wave equation for a wave on string	$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$
(2) Apply the boundary conditions @ x = 0 and x = l and obtain an appropriate wave function	$$y_n(x,t) = (A_n \cos\omega_n t + B_n \sin\omega_n t)\left(\sin\frac{\omega_n x}{c}\right)$$
(3) Obtain the Fourier sine series of displacement and velocity	$$y_0(x) = \sum A_n \sin\frac{\omega_n x}{c}, v_0(x) = \sum \omega_n B_n \sin\frac{\omega_n x}{c}$$
(4) Apply the initial conditions at t = 0 of displacement and velocity for a specific shape of string, $y \neq 0$ and $v = dy/dt = 0$ **(stationary plucked string),** Obtain A_n, $\omega_n B_n$ and the Fourier sine series of the specific shape of string  $y_0(x) = \frac{2dx}{l} \quad 0 \le x \le \frac{l}{2}$; $= \frac{2d(l-x)}{l} \quad \frac{l}{2} \le x \le l$	$$y_0(x) = \sum_{n\ \mathrm{odd}} A_n \sin\frac{\omega_n x}{c} = \sum_{n\ \mathrm{odd}} \left(\frac{8d}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{\omega_n x}{c}$$
(5) Obtain the energy	$$E = \sum E_n = \frac{2mc^2 d^2}{l^2} = \frac{2Td^2}{l}$$

Fourier series analysis of a rectangular velocity pulse on a string

- A string of length l, fixed at both ends, is struck by a mallet of width a about its center point.
- The initial condition at the moment of impact in terms of displacement and velocity are given as

$$y_0(x) = 0$$

$$v_0(x) = \frac{\partial y_0(x)}{\partial t} = 0 \quad \text{for} \quad \left|x - \frac{l}{2}\right| \geq \frac{a}{2}$$

$$= v \quad \text{for} \quad \boxed{\left|x - \frac{l}{2}\right| < \frac{a}{2}}$$

$$\boxed{\frac{l}{2} - \frac{a}{2} < x < \frac{l}{2} + \frac{a}{2}}$$

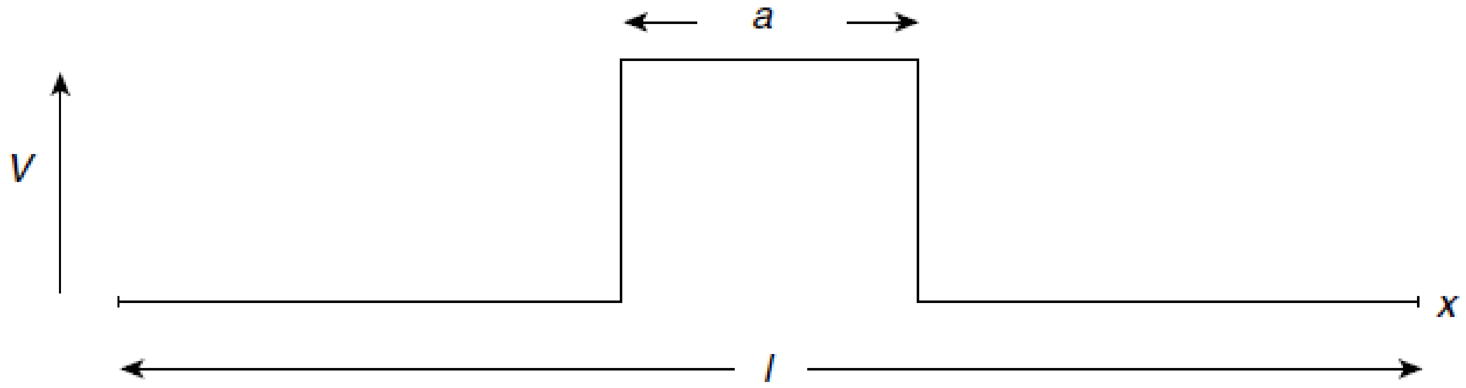


Figure 10.6 Velocity distribution at time $t = 0$ of a string length l, fixed at both ends and struck about its centre point by a mallet of width a. Displacement $y_0(x) = 0$; velocity $v_0(x) = v$ for $|x - l/2| < a/2$ and zero outside this region

- Recall the previous situation of the plucked string, arbitrary constants A_n and B_n can be found from

$$A_n = \frac{2}{l}\int_0^l y_0(x)\sin\frac{\omega_n x}{c}\,\mathrm{d}x$$

Note: we can start solving the problem from step 4 according to the summary table

$$\omega_n B_n = \frac{2}{l}\int_0^l v_0(x)\sin\frac{\omega_n x}{c}\,\mathrm{d}x$$

- However, in this case, A_n's are zero. **(WHY?)**
- This gives

$$v_0(x) = \sum_n \dot{y}_n = \sum_n \omega_n B_n \sin\frac{\omega_n x}{c}$$

- Therefore,

$$\omega_n B_n = \frac{2}{l}\int_{+l/2-a/2}^{l/2+a/2} v\sin\frac{\omega_n x}{c}\,\mathrm{d}x$$

$$\boxed{= \frac{4v}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi a}{2l}}$$

- Only n odd is considered. Thus, $$v_0(x) = \sum_{n\,\mathrm{odd}} \frac{4v}{n\pi} \sin\frac{n\pi a}{2l} \sin\frac{\omega_n x}{c}$$

- The total energy of vibration is found from the summation of energy per mode of oscillation.

$$E_n = \frac{4mv^2c^2}{l^2\omega_n^2} \sin^2\frac{\omega_n a}{2c}$$; $\because \omega_n = \dfrac{n\pi c}{l}$

- **The expression shows that again the energy of *n*th mode is proportional to n^{-2} (because $\omega_n = n\pi c/l$) and decreasing with increasing harmonic frequency**.
- The energy of mode *n*th can be rewritten in terms of angular frequency ω_n as

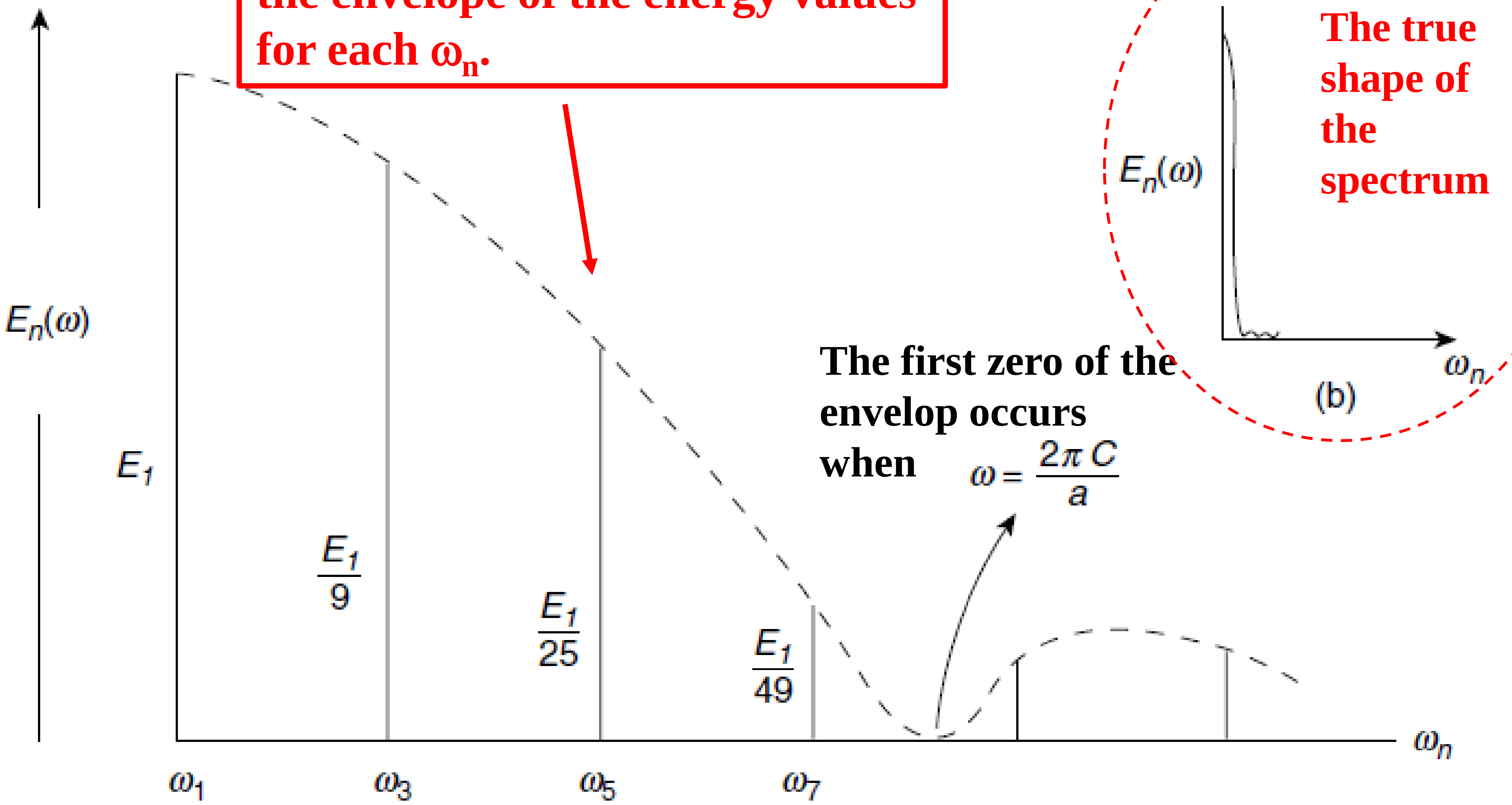
$$E_n(\omega) = \frac{mv^2a^2}{l^2}\frac{\sin^2(\omega_n a/2c)}{(\omega_n a/2c)^2}$$
$$\boxed{= \frac{mv^2a^2}{l^2}\frac{\sin^2\alpha}{\alpha^2}}$$

where $\alpha = \omega_n a/2c$

How does the distribution of the energy look like?

Energy spectrum of the harmonic ω_n

The curve of $\sin^2\alpha/\alpha^2$ appears as the envelope of the energy values for each ω_n.



(a)

The first zero of the envelop occurs when $\omega = \frac{2\pi C}{a}$

(b)

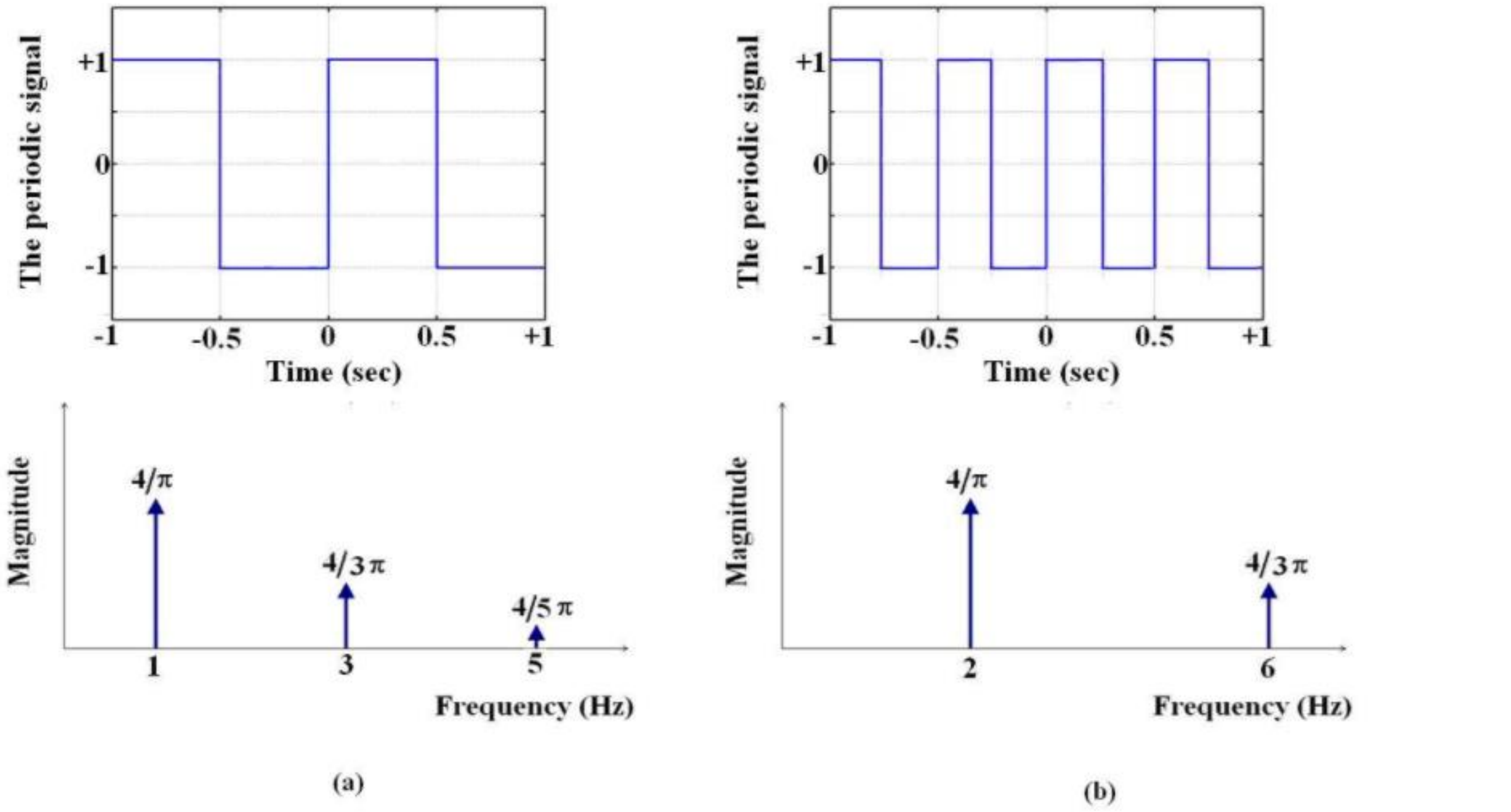
The true shape of the spectrum

What do we learn from the energy spectrum?

- The **major portion** of the energy in the velocity pulse is to be found in the low frequencies.
- The width (measured from the highest energy to the first zero of the energy) of the central frequency pulse contains most of the energy.
- This range of energy-bearing harmonics is known as the **spectral width** of the pulse given as

$$\Delta\omega \approx \frac{2\pi c}{a}$$

- For any spatial width $a = \Delta x$, the spectral width becomes $\Delta x \Delta\omega \approx 2\pi c$
- Because $\Delta t = \Delta x/c$, this gives $\Delta\omega\Delta t \approx 2\pi$ or $\Delta\nu\Delta t \approx 1$ knows as the **Bandwidth Theorem.**



(a) The periodic square wave with period of 1sec, and its corresponding spectrum,
(b) The square wave with period reduced to 0.5 second and its corresponding spectrum.
The relation just follows the bandwidth theorem , $\Delta\nu\Delta t \approx 1$.

Maybe we could hit YOUR BRAIN with our radiowave hammer…

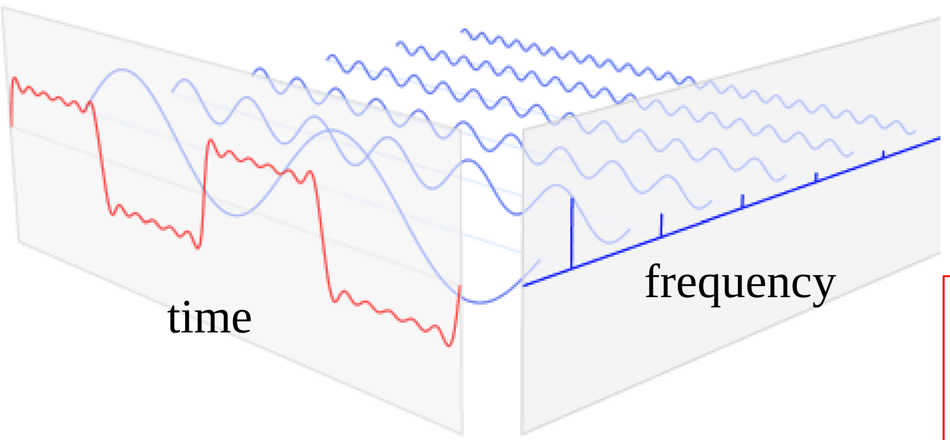
Applying the concept Fourier series analysis of disturbed wave to MRI

Magnetic Resonance Imaging is simply "listening" to the radiowaves emitted by hydrogen nuclei in the body, after hitting them with a radiowave hammer.

http://slideplayer.com/slide/8408101/26/images/26/Maybe+we+could+hit+YOUR+BRAIN+with+our+radiowave+hammer%E2%80%A6.jpg

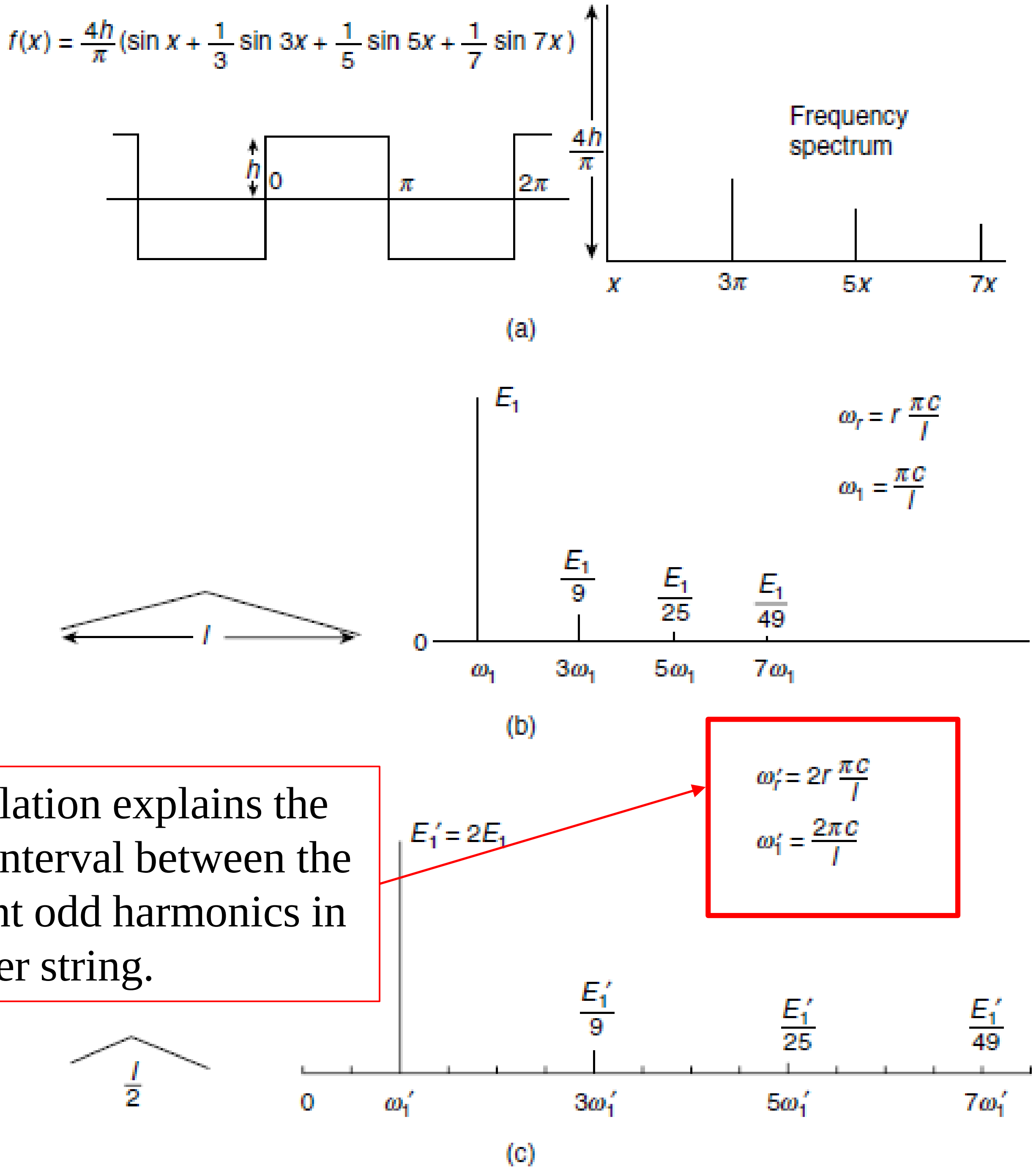
The frequency spectrum of Fourier series

- The Fourier series can always be represented as a frequency spectrum.



https://tex.stackexchange.com/questions/127375/replicate-the-fourier-transform-time-frequency-domains-correspondence-illustrati%3E

This relation explains the wider interval between the adjacent odd harmonics in a shorter string.



Fourier series to Fourier Integral (1)

- Recall the Fourier representation in the exponential form

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}nx}$$

where $2d_n = a_n - \mathrm{i}b_n (n \geq 0)$ and $2d_n = a_{-n} + \mathrm{i}b_{-n} (n < 0)$.

- Using the time as a variable, the Fourier representation is rewritten as

$$f(t) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}n\omega t}$$

- The coefficient d_n can be found from

$$d_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{e}^{-\mathrm{i}n\omega t}\,\mathrm{d}t$$

Derive this!

- Given $\omega = 2\pi\nu_1$ and T is the period, the Fourier series becomes

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f(t')\,\mathrm{e}^{-\mathrm{i}2\pi n\nu_1 t'}\,\mathrm{d}t'\right]\mathrm{e}^{\mathrm{i}2\pi n\nu_1 t}\cdot\frac{1}{T}$$

Fourier series to Fourier Integral (2)

- The conversion of the Fourier series to Fourier integral is based on following assumptions :

(1) The period T approaches infinity,

(2) The frequency $\nu_1 = 1/T \rightarrow 0$ and 1/T becomes infinitesimal and may be written as $d\nu$,

(3) $n\nu_1 = \nu$,

(4) The unit change in *n* becomes an infinitesimal change.

- The Fourier integral is therefore written as

$$f(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t')\, e^{-i2\pi\nu t'}\, dt' \right] e^{i2\pi\nu t}\, d\nu$$

Fourier integral and Fourier transform

- When $f(t)$ is non-periodic, the infinite number of frequency components in the integral form (not the sum) can be written as

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\, e^{i2\pi\nu t}\, d\nu$$

- Where

$$F(\nu) = \int_{-\infty}^{\infty} f(t')\, e^{-i2\pi\nu t'}\, dt'$$

is called Fourier transform of $f(t)$.

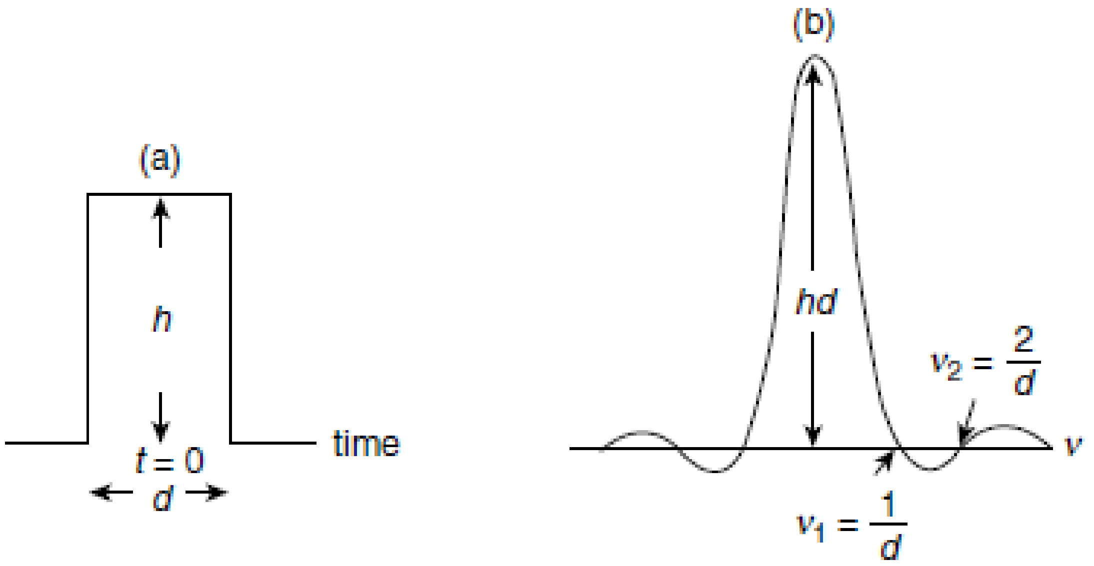
- This shows that integration with respect to one variable produces a function of the other.
- Both variable forms **Fourier pair of transform** and the product is non dimensional.

Example of Fourier transform : slit function

- Function of narrow slit extending *d* in time and of height *h*

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt = \int_{-d/2}^{d/2} h\, e^{-i2\pi\nu t}\, dt$$
$$= \frac{-h}{i2\pi\nu}\left[e^{-i2\pi\nu d/2} - e^{+i2\pi\nu d/2}\right] = hd\frac{\sin\alpha}{\alpha}$$

$$f(t) = \begin{cases} h & \text{for} \quad |t| < d/2 \\ 0 & \text{for} \quad |t| > d/2 \end{cases}$$



Example of Fourier transform : The Gaussian curve

- The Gaussian function of height h is symmetrically centered at t =0 is given by

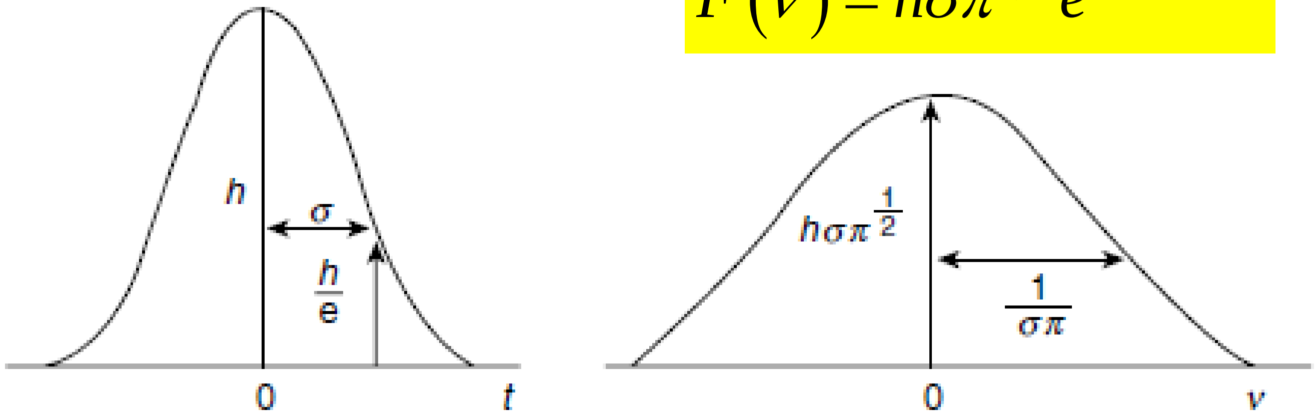
$$f\left(t\right)=he^{-t^2/\sigma^2}$$

- σ is the width parameter.

$$\begin{aligned}F(\nu) &= \int_{-\infty}^{\infty} h\,\mathrm{e}^{-t/\sigma^2}\,\mathrm{e}^{-\mathrm{i}2\pi\nu t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} h\,\mathrm{e}^{(-t/\sigma^2-\mathrm{i}2\pi\nu t+\pi^2\nu^2\sigma^2)}\,\mathrm{e}^{-\pi^2\nu^2\sigma^2}\,\mathrm{d}t \\ &= h\,\mathrm{e}^{(-\pi^2\nu^2\sigma^2)}\int_{-\infty}^{\infty}\mathrm{e}^{-(t/\sigma+\mathrm{i}\pi\nu\sigma)^2}\,\mathrm{d}t\end{aligned}$$

Given $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$

$$F\left(\nu\right)=h\sigma\pi^{1/2}e^{-\pi^2\nu^2\sigma^2}$$



(a) (b)

normalized Gaussian function

$$\boxed{\frac{1}{\left(\sigma\pi^{1/2}\right)}}\int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\,dt = 1$$

Height of the function

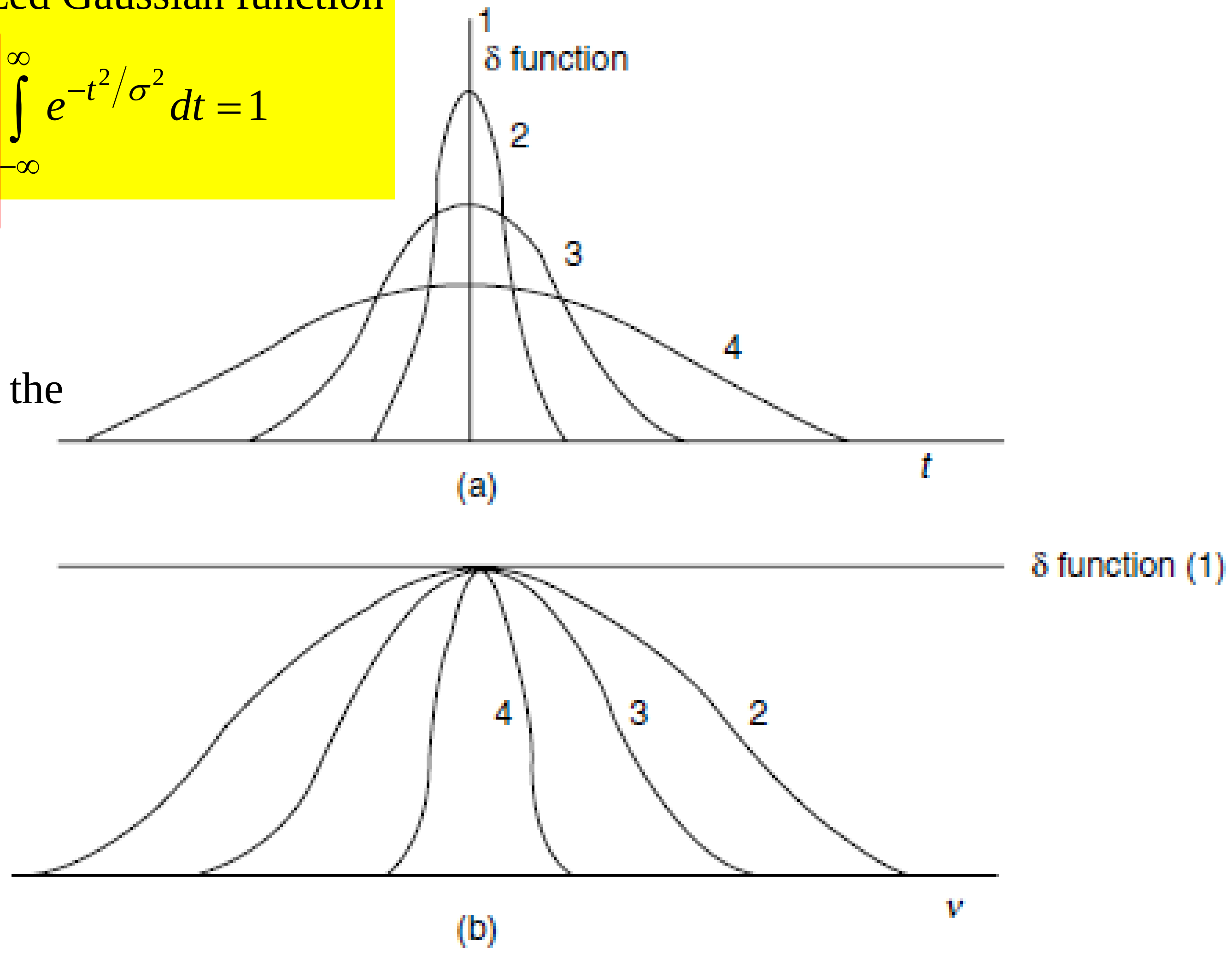


Figure 10.12 (a) A family of normalized Gaussian functions narrowed in the limit to Dirac's delta function; (b) the family of their Fourier transforms

- As height h → 0, the width σ → 0.
- The normalized Gaussian function becomes infinitely narrow which defines the Dirac delta (δ) function.
- The transform covers an increasing frequency components.

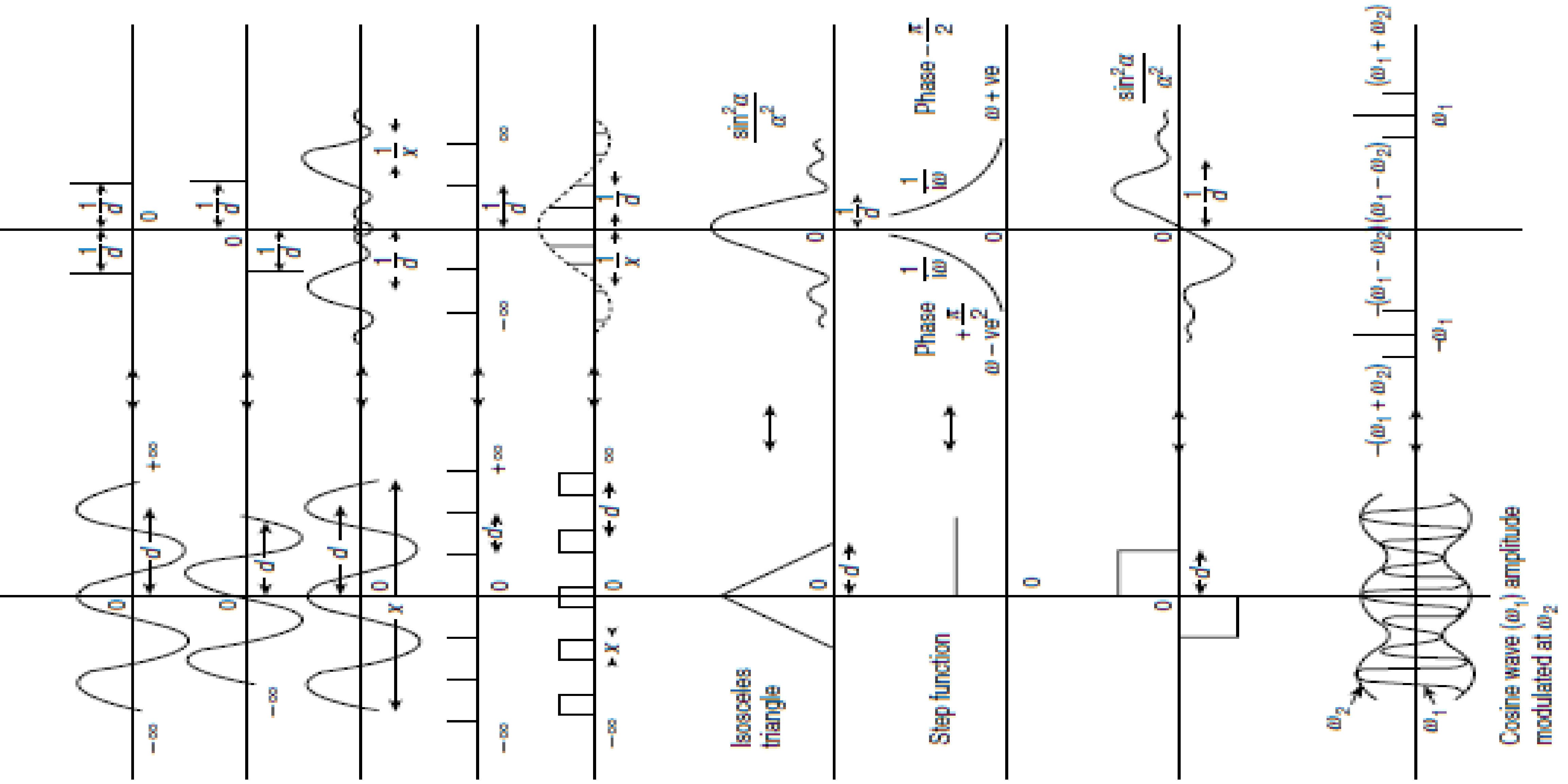


Figure 10.13 Some common Fourier transform pairs

Homework #11

10.3, 10.4